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An invariant renormalisation method for non-linear realisations of dynamical symmetries

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Abstract. The structure of the ultraviolet divergence is investigated for the field theoretical models with non-linear realisation of an arbitrary semi-simple Lie group, with spontaneously broken symmetry of vacuum. An invariant formulation of the background-field method of renormalisation is proposed which gives the manifestly invariant counter-terms off mass shell. A simple algorithm for construction of counter-terms is developed. It is based on invariants of the group of dynamical symmetry in terms of the Cartan forms. The results of one-loop and two-loop calculations are reported.

1. Introduction

There is a close connection between the initial symmetry of the classical field theory and the dynamics of its quantisation. The principle of symmetry itself often plays the role of a starting point for choosing a classical Lagrangian, and quantisation as well as renormalisation schemes are constructed which do not break the symmetry. Moreover, there is an opinion that taking into account the dynamical symmetry in theories with non-linear realisations, which are formally non-renormalisable, can provide renormalisability (Faddeev and Slavnov 1971) or, at least, various relations between counter-terms reflecting the dynamical effects. For instance, in two-dimensional models (Bardeen *et al* 1976) there arises the dynamical restoration of spontaneously broken symmetry of vacuum (quark confinement is considered to be connected with the analogous phenomenon in quantum chromodynamics (Bardeen and Pearson 1976)).

The present paper is devoted to the investigation of the structure of ultraviolet divergences in the field-theoretical models, which are the non-linear realisations of an arbitrary semi-simple Lie group with spontaneously broken symmetry of vacuum. A simple method is proposed for calculating the multi-loop counter-terms. With its help general formulae for one-loop and two-loop counter-terms are obtained.

Note that for such theories the calculation of two-loop counter-terms in the framework of standard perturbation theory cannot be performed because of the technical difficulties arising from the non-polynomial structure of the Lagrangian. The background-field method (De Witt 1967, Honerkamp 1972a, 't Hooft 1973a, Arefieva *et al* 1974, Tamura 1975, Grisaru *et al* 1975) which simplifies essentially the calculations of one-loop counter-terms, is not invariant off mass shell in the case of non-linear realisations. This fact makes the background-field method non-applicable for calculations of higher counter-terms.

A proposed modification of the background-field method is based on the theory of non-linear realisations of semi-simple Lie groups (Cartan 1927, 1946, Coleman *et al* 1968, Volkov 1973). This theory is operating with the images of quotient space G/H, where G is the initial group of dynamical symmetry and H is the subgroup of vacuum stability. By the phenomenological Lagrangian method (Cartan 1927, 1946, Coleman *et al* 1968, Volkov 1973), the algorithm for construction of the Lagrangians consists of the identification of the quotient space parameters with the fields of Goldstone particles and in the determination of the invariants of the group G defined on its quotient space.

The main point is to take into account the geometry of curved space of fields when separating the variables into background and quantum ones. We use the operation of addition of vectors on quotient space† and give an algorithm for counter-term construction on the basis of group invariants in terms of Cartan forms. Counter-terms obtained in this manner are invariant off mass shell (Kazakov *et al* 1977).

The paper is organised as follows. In § 2 the main conceptions of the phenomenological Lagrangian method are given and the Cartan forms are introduced. Section 3 is devoted to the method of renormalisation. A proposed modification of the background-field method is described and its invariance is proved. In § 4 the one-loop counter-terms are obtained and a general algorithm for counter-term construction is proposed. We use the algorithm developed in § 5 to obtain a general formula for two-loop counter-terms. In § 6 some conclusions and possible applications of the proposed formalism are presented.

2. Classical theory

The construction of non-linear realisations and, on the basis of these, of the invariants defining the structure of the phenomenological Lagrangian for an arbitrary group of dynamical symmetry, can be carried out by a standard procedure (Volkov 1973).

Let G be a $(k+r)$-parameter semi-simple symmetry group which degenerates the vacuum and produces the Goldstone particles; let H be its maximal subgroup leaving the vacuum invariant. The classical Lagrangian invariant under the group G has the form

$$\mathscr{L}(A)=\frac{1}{2C_2}\,\mathrm{Sp}\,\omega_\mu(A)\omega_\mu(A), \tag{1}$$

where C_2 is a quadratic Casimir operator of the group G, $\omega_\mu(A)$ are the differential Cartan forms, which can be defined via the finite transformations of the group G by the equation

$$\begin{gathered}G^{-1}(A)\partial_\mu G(A)=\mathrm{i}[\omega_\mu(A)+\theta_\mu(A)],\\ \omega_\mu(A)=\omega^i_\mu(A)X_i, \qquad \theta_\mu(A)=\theta^\alpha_\mu(A)Y_\alpha,\end{gathered} \tag{2}$$

where $Y_\alpha(\alpha=1,2,\ldots,\Gamma)$ are the generators of the subgroup H, $X_i(i=1,2,\ldots,k)$ are the generators of the coset G/H which complements H to the whole group G,

† Such an operation of addition was proposed for SU(2)×SU(2) theory by Honerkamp (1972b) and Ecker and Honerkamp (1973) and was developed in detail on the basis of Cartan theory for arbitrary dynamical groups (and among them quantum gravity) in papers of one of the authors (Pervushin 1975, 1976).

with the following algebra:

$$[Y_\alpha, Y_\beta] = \mathrm{i}A^\gamma_{\alpha\beta}Y_\gamma, \qquad [X_i, Y_\alpha] = \mathrm{i}B^k_{i\alpha}X_k, \qquad [X_i, X_k] = \mathrm{i}C^\alpha_{ik}Y_\alpha.$$

The group parameters (A) are identified with the fields of Goldstone particles, and the forms ω_μ and θ_μ have a simple geometrical meaning. The form ω_μ is determined with respect to some basis, components of an infinitesimal displacement $\mathrm{d}A$ from a point A to a point $A+\mathrm{d}A$ and the forms θ_μ define a change of the basis and are used to determine the covariant differentiation

$$D_\mu\omega_\nu = D_\nu\omega_\mu = \partial_\mu\omega_\nu + \mathrm{i}[\theta_\mu, \omega_\nu]. \tag{4}$$

The Cartan forms $\omega_\mu(A)$ and $\theta_\mu(A)$ are connected by the structure equations of the quotient space

$$\begin{aligned} &\partial_\mu\theta_\nu - \partial_\nu\theta_\mu + \mathrm{i}[\theta_\mu, \theta_\nu] = -\mathrm{i}[\omega_\mu, \omega_\nu] \equiv C_{\mu\nu}, \\ &\partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \mathrm{i}[\theta_\mu, \omega_\nu] = -\mathrm{i}[\omega_\mu, \theta_\nu] \equiv \hat{C}_{\mu\nu}. \end{aligned} \tag{5}$$

The explicit expression of the Cartan forms can be determined in the chosen parametrisation from equation (2) as a solution of the fundamental Cartan equations (see appendix 1).

The Lagrangian (1) is the invariant of the group G with minimal number of derivatives, and an arbitrary invariant can be expressed in terms of the Cartan forms in the following manner (Volkov *et al* 1973):

$$\mathrm{Sp}(D^{L_1}\ldots\omega[D^{L_2}\ldots\omega, [\ldots,[D^{L_{n-1}}\omega, D^{L_n}\ldots\omega]\ldots]]), \tag{6}$$

where L_i are the powers of the covariant differentials, points stand for the Lorentz indices over which summation is made. The invariants (6) form a complete set.

3. A method of renormalisation

We shall construct the renormalisation procedure on the basis of the invaraint formulation of the background field method. An excellent description of the background-field method of renormalisation can be found in the original paper by 't Hooft (1973a). Therefore we recapitulate only the main points of the method and describe the proposed modification for the case of non-linear realisations.

Let us consider the field theory with the Lagrangian $\mathscr{L}(A)$. The change of variables

$$A \to A + \phi \tag{7}$$

is to be carried out, where the field A is called the background (or classical or external) one and the field ϕ is called the quantum (or internal) one. Then the generating functional for the loop diagrams is

$$F[A] = \frac{1}{N}\int \prod_x \delta\mu(\phi)\exp\left[\mathrm{i}\int \mathrm{d}x\left(\mathscr{L}(A+\phi) - \mathscr{L}(A) - \phi\frac{\delta\mathscr{L}(A)}{\delta A}\right)\right]. \tag{8}$$

Counter-terms in the Lagrangian are obtained by expanding $\mathscr{L}(A+\phi)$ into the Taylor series in ϕ. To construct the counter-terms of a given order we need only the finite number of the expansion terms. (To remove the divergences in the subgraphs the same expansion should be made in the counter-terms of lower order which, for this purpose, should be known off mass shell.)

If the Lagrangian possesses some kind of linear symmetry it is easy to show that the generating functional (8) is also invariant. The counter-terms obtained automatically satisfy all the Ward identities.

However, in the case of non-linear realisations, the replacement (7) breaks the initial symmetry group. The generating functional (8) no longer leads to the invariant counter-terms off mass shell. Therefore we propose another method for separation of the background fields. The idea is the following (see Pervushin 1975).

Let the Lagrangian be invariant under the group G of the field transformations

$$G(A')=G(g)G(A), \tag{9}$$

where $G(g)$ is the transformation of the group G and

$$\mathscr{L}(G(A'))=\mathscr{L}(G(A)). \tag{10}$$

The transformations (9) define the non-linear realisation of the group on the coordinates of space of the particles A. Then the natural way for separating the classical fields without violating the symmetry is to use the geometric properties of the group space of fields, i.e. to understand the sum of vectors (7) as the addition of vectors in the curved space (addition of vectors in the quotient space G/H), i.e.

$$G(A)\to G(A)G(\phi), \qquad A\to A(+)\phi. \tag{11}$$

For such an 'addition' of fields the 'sum' is an element of the same space and has the same transformation properties under the group G:

$$G(A')G(\phi')=G(g)G(A)G(\phi). \tag{12}$$

From equations (9), (12) it follows that both the Lagrangian and all its Taylor series are invariant under the group G. We show this for the first variation of the Lagrangian representing it in the form

$$\frac{\delta\mathscr{L}(A(+)\phi)}{\delta\phi}\bigg|_{\phi=0}=\frac{\delta\mathscr{L}(A)}{\delta A}=\frac{\delta\mathscr{L}(G_AG_\phi)}{\delta(G_AG_\phi)}\frac{\delta(G_AG_\phi)}{\delta G_\phi}\frac{\delta G_\phi}{\delta\phi}\bigg|_{\phi=0}=\frac{\delta\mathscr{L}(G_A)}{\delta G_A}G_A\frac{\delta G_\phi}{\delta\phi}\bigg|_{\phi=0}.$$

Then we have:

$$\frac{\delta\mathscr{L}(A')}{\delta A'}=\frac{\delta\mathscr{L}(G_{A'})}{\delta G_{A'}}G_{A'}\frac{\delta G_{\phi'}}{\delta_{\phi'}}\bigg|_{\phi'=0}$$

$$=\frac{\delta\mathscr{L}(G_{A'})}{\delta G_A}\frac{\delta G_A}{\delta G_{A'}}G_{A'}\frac{\delta G_{\phi'}}{\delta\phi'}\bigg|_{\phi'=0}=\frac{\delta\mathscr{L}(G_A)}{\delta G_A}G_g^{-1}G_gG_A\frac{\delta G_\phi}{\delta\phi}\bigg|_{\phi=0}=\frac{\delta\mathscr{L}(A)}{\delta A}.$$

Here we used the formulae (9), (10) and the fact that $\delta G_\phi/\delta\phi|_{\phi=0}=$ constant and is not transformed. Hence, taking into account the invariance of the integration measure $\delta\mu(\phi)$, we obtain

$$F[A']=F[A].$$

Thus the use of addition law (11) enables us to construct an explicitly invariant background formalism in the case of non-linear realisations. We do not here use the equations of motion for the classical fields, which are important for the construction of finite Green functions off mass shell.

The transformation (11) has a simple geometrical interpretation. It is a shift of the origin to the point A, which corresponds to the transformation of quantum fields with the parameters being the classical ones. Cartan forms in the new coordinates could be found substituting (11) into (2):

$$[G(A)G(\phi)]^{-1}\partial_\mu[G(A)G(\phi)] = \mathrm{i}[\bar{\omega}_\mu(A,\phi)+\bar{\theta}_\mu(A,\phi)], \tag{13}$$

where the explicit form of $\bar{\omega}^i_\mu(A,\phi)$ and $\bar{\theta}^i_\mu(A,\phi)$ in the chosen parametrisation is defined as a solution of the Cartan fundamental equations with the non-zero boundary conditions $\bar{\omega}_\mu(A,0)=\omega_\mu(A)$, $\bar{\theta}_\mu(A,0)=\theta_\mu(A)$ (see appendix 1). They are

$$\bar{\omega}^i_\mu(A,\phi)=\sum_{n=0}^{\infty}(-1)^n(\mathcal{M}^n_\phi)^i_l\left(\frac{\omega^l_\mu(A)}{(2n)!}+\frac{(D_\mu\phi)^l}{(2n+1)!}\right), \tag{14}$$

$$\bar{\theta}^i_\mu(A,\phi)=\sum_{n=0}^{\infty}(-1)^n\phi^j C^i_{jk}(\mathcal{M}^n_\phi)^k_l\left(\frac{\omega^l_\mu(A)}{(2n+1)!}+\frac{(D_\mu\phi)^l}{(2n+2)!}\right),$$

where $(D_\mu\phi)^l=\partial_\mu\phi^l+\mathrm{i}(\theta_\mu(A)\phi)^l$ is the covaraint derivative of the field ϕ.

Then for the theory with the Lagrangian (1) we have

$$\mathscr{L}(A(+)\phi)=\frac{1}{2C_2}\operatorname{Sp}\bar{\omega}_\mu(A,\phi)\bar{\omega}_\mu(A,\phi). \tag{15}$$

To find the counter-terms $\Delta\mathscr{L}_n$ in the n-loop approximation we have first to expand the Lagrangian over the quantum field up to ϕ^{2n} and then to carry out the expansion of $\Delta\mathscr{L}_{n-1}$, up to ϕ^{2n-2}, changing ω_μ, θ_μ by $\bar{\omega}_\mu$, $\bar{\theta}_\mu$. The n-loop expansion of $\Delta\mathscr{L}_{n-1}$ reproduces the subtraction in the subgraphs.

4. The algorithm for construction of counter-terms

The proposed formalism enables us to develop a simple algorithm for construction of counter-terms in any approximation. From formula (14) it follows that all coefficient functions in the expansion of the Lagrangian (15) are products of forms $\omega_\mu(A)$ and $\theta_\mu(A)$. Furthermore, only three types of external structures, namely, $\omega_\mu(A)\omega_\mu(A)$, $D_\mu(A)\omega_\mu(A)$ and $D_\mu(A)D_\mu(A)$ exist in all orders. This enables us to obtain manifestly invariant counter-terms, which are written in terms of the Cartan forms without expansion over fields. They are constructed from the set of linearly independent invariants of the symmetry group (6). From the analysis of the divergent diagrams it follows that in the n-loop approximation counter-terms are uniform functions of the Cartan forms of the power $[D\omega]^{2k}[\omega]^{2(n+1-2k)}$, $k=0,1,\ldots,\{\frac{1}{2}n\}$.†

Let us consider first the one-loop approximation. By formulae (8), (14) and (15) the generating Lagrangian for one-loop diagrams has the form

$$\mathscr{L}^{(1)}_{\text{int}}(A,\phi)=\tfrac{1}{2}[(D_\mu\phi)^i(D_\mu\phi)^i-\phi\omega_\mu\omega_\mu\phi]. \tag{16}$$

Here the generators of the group X_i and Y_α are chosen in the adjoint representation, and for simplicity we put them equal to the structure constants A, B and C in formula (3). We obtain the types of vertices shown in figure 1. Hereafter the single internal

† This fact results from the application of the continuous dimensional regularisation method. In another case the power pointed out should be maximal for the invariants at the n-loop approximation.

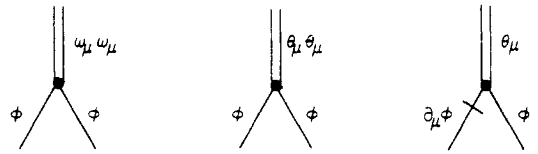

Figure 1. The types of one-loop vertices.

lines on the diagrams denote the quantum fields ϕ and the double external ones denote the forms of classical fields A.

Linearly independent invariants of power four have the form

$$I_1 = \mathrm{Sp}\ \omega_\mu\omega_\mu\omega_\nu\omega_\nu, \qquad I_2 = \mathrm{Sp}\ \omega_\mu\omega_\nu\omega_\mu\omega_\nu. \tag{17}$$

However, the invariant I_2 cannot be directly reproduced by the one-loop diagrams with the vertices presented in figure 1. Therefore, we recompose the invariants I_1 and I_2 using the structure equations (5)

$$\begin{aligned} J_1 &= I_1 - I_2 = \mathrm{i}\ \mathrm{Sp}\ \omega_\mu C_{\mu\nu}\omega_\nu = \tfrac{1}{2}\mathrm{Sp}\ C_{\mu\nu}C_{\mu\nu}, \\ J_2 &= I_1 + I_2 = \mathrm{Sp}(\omega_\mu\omega_\mu\omega_\nu\omega_\nu + \omega_\mu\omega_\nu\omega_\mu\omega_\nu). \end{aligned} \tag{18}$$

Note that new invariants include the structures which are directly reproduced by one-loop diagrams and are included only in one of them. We shall call these structures the characteristic ones.

They are

$$\begin{aligned} J_1 &\Rightarrow \mathrm{Sp}\ \theta_\mu\theta_\nu\theta_\mu\theta_\nu, \\ J_2 &\Rightarrow \mathrm{Sp}\ \omega_\mu\omega_\mu\omega_\nu\omega_\nu. \end{aligned} \tag{19}$$

Hence we shall look for the counter-terms in the form

$$\Delta\mathscr{L}_1 = b_1 J_1 + b_2 J_2, \tag{20}$$

where the coefficients b_1 and b_2 are determined by the contribution to characteristic structures (19) from different divergent diagrams. For this purpose it is sufficient to examine two diagrams of figure 2 where $\epsilon = (d-4)/2$ is the parameter of dimensional regularisation, $d \to 4$ is the space–time dimension. Substituting $b_1 = [12(16\pi^2\epsilon)]^{-1}$ and $b_2 = [4(16\pi^2\epsilon)]^{-1}$ into (20) and taking into account (18) we obtain:

$$\Delta\mathscr{L}_1 = \frac{1}{3(16\pi^2\epsilon)}(I_1 + \tfrac{1}{2}I_2). \tag{21}$$

Now we can formulate a general algorithm for counter-term construction in the n-loop approximation. Counter-terms are constructed in the form

$$\Delta\mathscr{L}_n = a_1 I_1 + \ldots + a_N I_N, \tag{22}$$

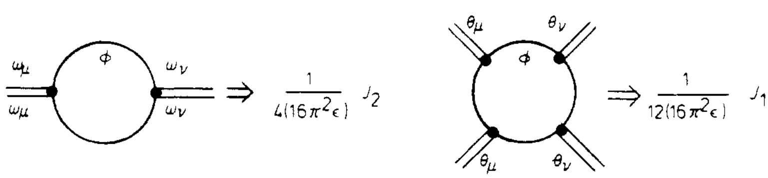

Figure 2. The diagrams contributing to the one-loop counter-terms.

where $I_1, \ldots, I_N$ is the complete set of linearly independent invariants (6) and $a_1, \ldots, a_N$ are functions of the regularisation parameter. To determine a_i it is necessary:

(i) to write down the generating functional for the n-loop approximation taking into account equations (8), (14), (15) and to expand the lower counter-terms with the help of equation (14);

(ii) to choose the complete sets of linearly independent invariants of the required power over Cartan forms, using equation (6) and the structure equations (5);

(iii) to recompose the invariants in the form where they are directly reproduced by the combinations of the coefficient functions from the generating Lagrangian;

(iv) to choose the characteristic structure in every new invariant and to calculate the contribution to it from the divergent diagrams of the n-loop approximation.

Let us demonstrate the application of the proposed algorithm to the determination of the two-loop counter-terms.

5. Two-loop approximation

The generating Lagrangian in the two-loop approximation is

$$\begin{aligned}\mathscr{L}_{\mathrm{int}}^{(2)} = {} & \tfrac{1}{2}[(D_\mu\phi)^i(D_\mu\phi)^i - \phi\omega_\mu\omega_\mu\phi] \\ & + \tfrac{1}{2}[-\tfrac{4}{3}\phi(D_\mu\phi)\omega_\mu\phi - \tfrac{1}{3}\phi(D_\mu\phi)(D_\mu\phi)\phi + \tfrac{1}{3}\phi X^i\omega_\mu\phi\phi X^i\omega_\mu\phi].\end{aligned} \tag{23}$$

We choose linearly independent invariants in the following manner:

$$\begin{aligned} & I_1 = \mathrm{Sp}\ \omega_\mu\omega_\mu\omega_\nu\omega_\nu\omega_\rho\omega_\rho, && I_2 = \mathrm{Sp}\ \omega_\mu\omega_\nu\omega_\mu\omega_\nu\omega_\rho\omega_\rho, \\ & I_3 = \mathrm{Sp}\ \omega_\mu\omega_\nu\omega_\nu\omega_\mu\omega_\rho\omega_\rho, && I_4 = \mathrm{Sp}\ \omega_\mu\omega_\nu\omega_\rho\omega_\mu\omega_\nu\omega_\rho, \\ & I_5 = \mathrm{Sp}\ \omega_\mu\omega_\nu\omega_\rho\omega_\mu\omega_\rho\omega_\nu, && \\ & I_6 = \mathrm{Sp}\ D_\mu\omega_\mu D_\nu\omega_\nu\omega_\rho\omega_\rho, && I_7 = \mathrm{Sp}\ D_\mu\omega_\nu D_\mu\omega_\nu\omega_\rho\omega_\rho, \\ & I_8 = \mathrm{Sp}\ D_\mu\omega_\mu\omega_\rho\omega_\nu\omega_\rho, && I_9 = \mathrm{Sp}\ D_\mu\omega_\nu D_\mu\omega_\rho\omega_\rho\omega_\nu, \\ & I_{10} = \mathrm{Sp}\ D_\mu\omega_\mu\omega_\rho D_\nu\omega_\nu\omega_\rho, && I_{11} = \mathrm{Sp}\ D_\mu\omega_\mu\omega_\nu D_\nu\omega_\rho\omega_\rho \end{aligned} \tag{24}$$

Let us recompose them taking into account the structure equations (5). Thus we obtain the new system of invariants.

$$\begin{aligned} & J_1 = I_2 - I_3 = \mathrm{i}\ \mathrm{Sp}\ \omega_\mu\omega_\nu C_{\mu\nu}\omega_\rho\omega_\rho = -\tfrac{1}{2}\mathrm{Sp}\ C_{\mu\nu}C_{\mu\nu}\omega_\rho\omega_\rho, \\ & J_2 = I_2 + I_3 = \mathrm{Sp}(\omega_\mu\omega_\nu\omega_\mu\omega_\nu\omega_\rho\omega_\rho + \omega_\mu\omega_\nu\omega_\nu\omega_\mu\omega_\rho\omega_\rho), \\ & J_3 = -I_1 + 3I_2 + I_4 - 3I_5 = -\mathrm{i}\ \mathrm{Sp}\ C_{\mu\nu}C_{\rho\mu}C_{\nu\rho}, \\ & J_4 = I_1 - I_2 + I_4 - I_5 = -\mathrm{Sp}(C_{\mu\nu}C_{\rho\mu}\omega_\nu\omega_\rho + C_{\mu\rho}C_{\rho\mu}\omega_\rho\omega_\nu), \end{aligned} \tag{25}$$

$$\begin{aligned} J_5 = {} & -I_1 - I_2 + I_4 + I_5 \\ & = \mathrm{i}\ \mathrm{Sp}\ (\omega_\mu\omega_\nu C_{\rho\mu}\omega_\nu\omega_\rho + \omega_\mu\omega_\nu C_{\rho\mu}\omega_\rho\omega_\nu + \omega_\nu\omega_\mu C_{\rho\mu}\omega_\nu\omega_\rho + \omega_\nu\omega_\mu C_{\rho\mu}\omega_\rho\omega_\nu), \end{aligned}$$

$$J_6 = I_6; \qquad J_7 = I_7, \qquad J_8 = I_8, \qquad J_9 = I_9, \qquad J_{10} = I_{10}, \qquad J_{11} = I_{11}.$$

We choose the characteristic structures in the invariants

$$
\begin{aligned}
&J_1 \Rightarrow \mathrm{Sp}\ \partial_\mu\theta_\nu\partial_\mu\theta_\nu\omega_\rho\omega_\rho, && J_6 \Rightarrow \mathrm{Sp}\ \partial_\mu\omega_\mu\partial_\mu\omega_\mu\partial_\nu\omega_\nu\omega_\rho\omega_\rho,\\
&J_2 \Rightarrow \mathrm{Sp}\ \omega_\mu\omega_\nu\omega_\mu\omega_\nu\omega_\rho\omega_\rho, && J_7 \Rightarrow \mathrm{Sp}\ \partial_\mu\omega_\nu\partial_\mu\omega_\nu\omega_\rho\omega_\rho,\\
&J_3 \Rightarrow \mathrm{Sp}\ \theta_\mu\theta_\nu\theta_\rho\theta_\mu\theta_\nu\theta_\rho, && J_8 \Rightarrow \mathrm{Sp}\ \partial_\mu\omega_\mu\partial_\nu\omega_\rho\omega_\nu\omega_\rho,\\
&J_4 \Rightarrow \mathrm{Sp}\ \partial_\mu\theta_\nu\partial_\rho\theta_\mu\omega_\omega\omega_\rho, && J_9 \Rightarrow \mathrm{Sp}\ \partial_\mu\omega_\nu\partial_\mu\omega_\rho\omega_\rho\omega_\nu\\
&J_5 \Rightarrow \mathrm{Sp}\ \omega_\mu\omega_\nu\partial_\rho\theta_\mu\omega_\nu\omega_\rho, && J_{10} \Rightarrow \mathrm{Sp}\ \partial_\mu\omega_\mu\omega_\rho\partial_\nu\omega_\nu\omega_\rho,\\
&J_{11} \Rightarrow \mathrm{Sp}\ \partial_\mu\omega_\mu\omega_\mu\omega_\nu\partial_\nu\omega_\rho\omega_\rho.
\end{aligned}
\tag{26}
$$

The diagrams shown in figure 3 contribute to these structures.

To evaluate the singular terms of the integrals corresponding to the diagrams presented in figure 3, and to perform the subtraction in the subgraphs, we used the renormalisation scheme proposed by 't Hooft (1973b). The transformations of the expressions obtained were performed by the formulae presented in appendix 2. As a final result, we have the following expression for the counter-terms of the two-loop approximation:

$$
\begin{aligned}
&\Delta\mathcal{L}_2 = \sum_{i=1}^{11} \frac{2C_2}{9(16\pi^2\epsilon)^2} b_i I_i,\\
&b_1 = -\frac{1}{48}\left(1 + 582 \cdot \frac{5}{N+2}\right) + \frac{\epsilon}{96\cdot 3}\left(71 - 458 \cdot \frac{5}{N+2}\right),\\
&b_2 = \frac{1}{24}\left(11 + 1056 \cdot \frac{5}{N+2}\right) - \frac{\epsilon}{96\cdot 6}\left(211 + 3008 \cdot \frac{5}{N+2}\right),\\
&b_3 = -\frac{1}{16}\left(7 - 50 \cdot \frac{5}{N+2}\right) + \frac{\epsilon}{96\cdot 2}\left(23 - 676 \cdot \frac{5}{N+2}\right),\\
&b_4 = -\frac{1}{48}\left(1 - 570 \cdot \frac{5}{N+2}\right) + \frac{\epsilon}{96\cdot 3}\left(71 + 358 \cdot \frac{5}{N+2}\right),\\
&b_5 = \frac{1}{48}\left(1 - 1818 \cdot \frac{5}{N+2}\right) - \frac{\epsilon}{96\cdot 3}\left(71 - 458 \cdot \frac{5}{N+2}\right),\\
&b_6 = -\frac{1}{12}\left(1 + \frac{5}{N+2}\right) + \frac{\epsilon}{18}\left(5 - \frac{7}{4} \cdot \frac{5}{N+2}\right),\\
&b_7 = \frac{3}{16} - \frac{\epsilon}{32}\left(1 + 8 \cdot \frac{5}{N+2}\right),\\
&b_8 = \frac{11}{27} \cdot \frac{5}{N+2}\epsilon, \qquad b_9 = -\frac{1}{16} + \frac{\epsilon}{16\cdot 9}\left(13 - 46 \cdot \frac{5}{N+2}\right),\\
&b_{10} = \frac{1}{12} \cdot \frac{5}{N+2} + \frac{7}{72} \cdot \frac{5}{N+2}\epsilon,\\
&b_{11} = -\frac{1}{16} + \frac{\epsilon}{16\cdot 54}\left(175 - 176 \cdot \frac{5}{N+2}\right),
\end{aligned}
\tag{27}
$$

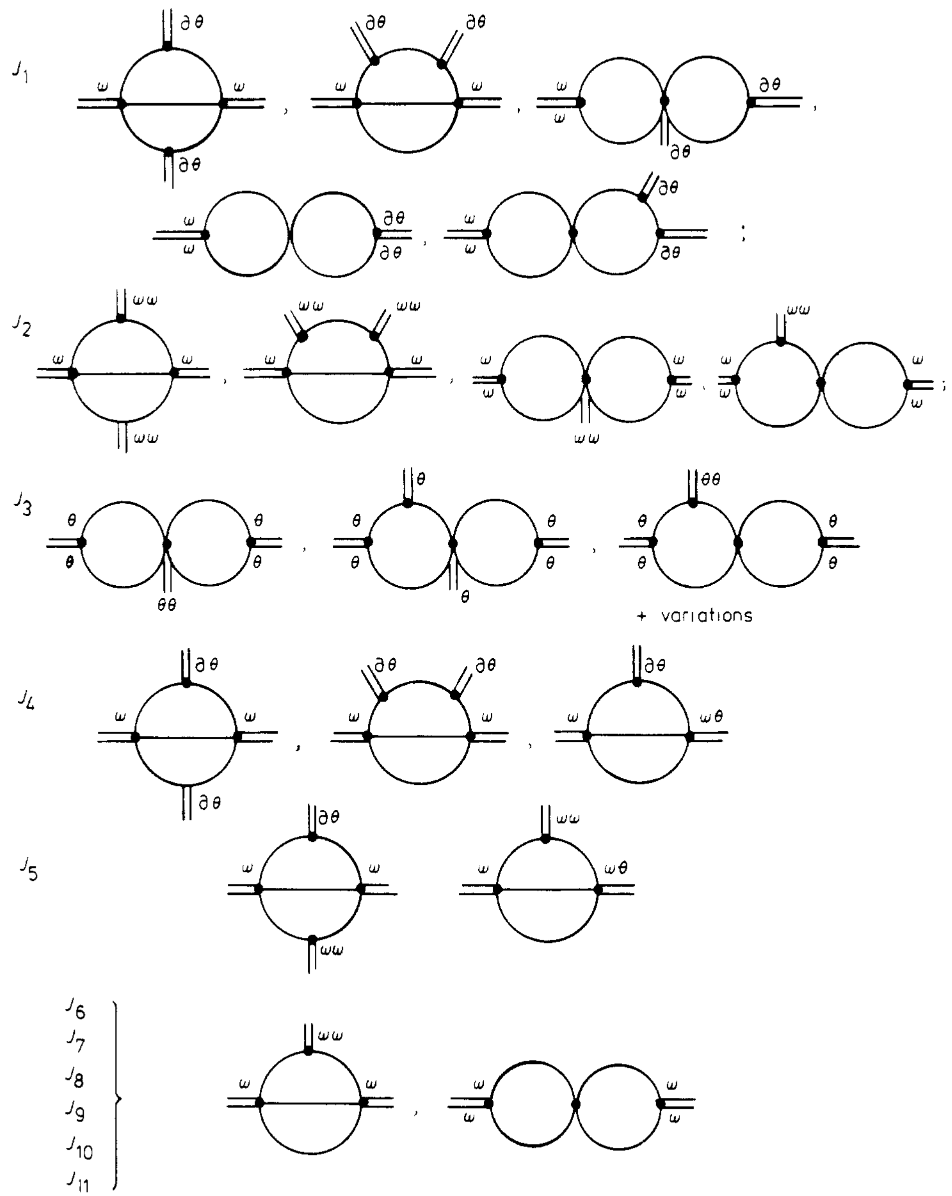

Figure 3. The diagrams contributing to the two-loop counter-terms.

where C_2 is the Casimir operator, N is the number of the group parameters (see appendix 2).

We point out once more that invariants (24) do not include the products of traces. This reflects the so-called property of algebraic duality (Volkov *et al* 1973). To our mind such products, even if they appear, should not contribute to the conter-terms. Otherwise, they should be included in the initial set of the invariants (6) and (24). This gives no principal change of the developed scheme, but only brings some technical difficulties.

6. Conclusion

In the present paper we have proposed an invariant approach to the problem of the ultraviolet divergences. The approach allows the most consistent consideration of the symmetry properties when the counter-terms are constructed for the field theories with non-linear realisation of an arbitrary semi-simple Lie group, with spontaneously broken symmetry of vacuum. We mention the advantages of the formalism developed.

(i) The counter-terms are manifestly invariant without taking into account the equations of motion. This enables us to investigate the structure of the divergences of the S-matrix and Green functions off mass shell.

(ii) The counter-terms are constructed out of a small number of foreknown invariants of the group.

(iii) The coefficient-determination procedure enables us to calculate the minimal possible number of diagrams.

(iv) All the calculations are performed directly in covariant terms of Cartan forms without expansion over the fields.

In considering the non-linear realisations, the question of their non-renormalisability naturally arises. The increase of power of invariants in the Cartan forms with the number of loops, apparently, indicates that the symmetry arguments alone do not lead to the closed form of the Lagrangian and to its renormalisability in the ordinary sense.

However, the situation is of interest when the coefficients for higher invariants are not arbitrary, but are determined by the coefficients for lower ones. Physically, it means the dynamical restoration of a symmetry group, which is wider than the initial one. The carrier of this symmetry would play the role of the bound state of initial fields. A phenomenon of this type takes place in two-dimensional models (Bardeen *et al* 1976). The attempts to expand it to the four-dimensional ones are known to be connected with the mechanism of quark confinement (Bardeen and Pearson 1976). It was shown (Tamura 1975) in the case of non-linear realisation of the chiral $SU(2)\times SU(2)$ group that the one-loop counter-terms cannot be interpreted as a dynamical appearance of an isoscalar σ-field. However, in the consideration of a more complicated variant of the σ-model, including, for instance, isotensor σ-fields the one-loop approximation can be interpreted in this manner. The question of its validity in the two-loop and higher approximations is now under consideration.

In conclusion we mention that the formalism proposed can be applied to the theories with an algebra different from (3), for instance to quantum gravity (Pervushin 1976, Borisov and Ogievetsky 1974). A treatment of quantum gravity in terms of a variant of non-linear realisation of dynamical affine symmetry (cf. Borisov and Ogievetsky 1974) is in preparation by the authors.

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Appendix 1. The fundamental Cartan equations

Here we find the Cartan forms in the exponential parametrisation for the finite group transformation (Pervushin 1975)

$$G(A)=\exp(\mathrm{i}X_aA^a).$$

This parametrisation corresponds to the normal coordinates in space of the Goldstone particles. We have

$$\exp(-\mathrm{i}X^aA^a)\partial_\mu\exp(\mathrm{i}X_aA^a)=\mathrm{i}[\omega^i_\mu(A)X_i+\theta^\alpha_\mu(A)Y_\alpha].$$

Let us introduce a parameter t by the substitution $A\to At$. We obtain

$$\exp(-\mathrm{i}X_aA^at)\partial_\mu\exp(\mathrm{i}X_aA^at)=\mathrm{i}[\omega^i_\mu(At)X_i+\theta^a_\mu(At)Y_\alpha].$$

Differentiating this equation with respect to t, we obtain the fundamental Cartan equations

$$\frac{\partial\omega^i_\mu}{\partial t}=\partial_\mu A^i+A^k\theta^\beta_\mu B^i_{k\beta},\qquad \frac{\partial\theta^i_\mu}{\partial t}=A^i\omega^l_\mu C^i_{jl}$$

with zero boundary conditions

$$\omega_\mu(0)=0,\qquad \theta_\mu(0)=0.$$

In the general case the solutions of the Cartan fundamental equations can be written as the series

$$\omega^i_\mu(A)=\sum_{n=0}^{\infty}(-1)^n(\mathcal{M}^n_A)^i_l\frac{\partial_\mu A^l}{(2n+1)!},$$

$$\theta^\alpha_\mu(A)=A^jC^\alpha_{jk}\sum_{n=0}^{\infty}(-1)^n(\mathcal{M}^n_A)^k_l\frac{\partial_\mu A^l}{(2n+2)!},$$

where

$$(\mathcal{M}^0_A)^i_l=\delta_{il},\qquad (\mathcal{M}_A)^i_l=-B^i_{k\beta}A^kC^\beta_{jl}A^j,\qquad (\mathcal{M}^2_A)^i_l=(\mathcal{M}_A)^i_k(\mathcal{M}_A)^k_l,$$

and $B^i_{k\beta}$, C^β_{jl} are the structure constants of the group. In the coordinate system (12) the forms $\bar{\omega}_\mu(A,\phi)$, $\bar{\theta}_\mu(A,\phi)$ can be found in the same way, by merely changing the boundary conditions:

$$\bar{\omega}_\mu(A,0)=\omega_\mu(A),\qquad \bar{\theta}_\mu(A,0)=\theta_\mu(A).$$

To illustrate how the algorithm works, we give the expressions for the Cartan forms for the SU(2)×SU(2) chiral group. In normal coordinates, they are

$$\omega^i_\mu(A)=\partial_\mu A^k\left[\delta_{ik}+\left(\delta_{ik}-\frac{A^iA^k}{A^2}\right)\left(\frac{\sin\sqrt{A^2}}{\sqrt{A^2}}-1\right)\right]$$

$$\theta^i_\mu(A)=-\epsilon^{ijk}A^j\partial_\mu A^k\frac{\cos\sqrt{A^2}-1}{A^2},\qquad i,j,k=1,2,3;\qquad C_2=2,N=3.$$

Appendix 2. Some useful formulae

For the generators of group G taken in the adjoint representation ${}_a(X^b)_c = \mathrm{i}f^{abc}$; $a, b, c = 1, 2, \ldots, N$; f^{abc} being the structure constant of the group, the following formulae are valid:

$f^{abc}f^{abd} = C_2\delta^{cd}$, $\qquad$ C_2 is a quadratic Casimir operator,

$X^aX^a = C_2 I, \qquad \mathrm{Sp}\, X^a = 0, \qquad \mathrm{Sp}\, X^aX^b = C_2\delta^{ab} \qquad \mathrm{Sp}\, X^aX^bX^c = \tfrac{1}{2}\mathrm{i}C_2 f^{abc},$

$$\mathrm{Sp}\, X^aX^bX^cX^d = \frac{5C_2^2}{6(N+2)}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) - \frac{C_2}{6}(f^{ab\cdot}f^{cd\cdot} + f^{da\cdot}f^{bc\cdot}),$$

$$\begin{aligned}
\mathrm{Sp}\, & X^aX^bX^cX^dX^e \\
&= \mathrm{i}\frac{5C_2^2}{12(N+2)}[\delta^{ab}f^{cde} + \delta^{ac}f^{bde} + \delta^{ad}f^{bce} + \delta^{ae}f^{bcd} + \delta^{be}f^{ade} \\
&\quad + \delta^{bd}f^{ace} + \delta^{be}f^{acd} + \delta^{cd} + \delta^{ce}f^{abe} + f^{abd} + \delta^{de}f^{abc}] \\
&\quad + \mathrm{i}\frac{C_2}{20}[f^{bc\cdot}f^{\cdot a}f^{\cdot de} + f^{ae\cdot}f^{\cdot b\cdot}f^{\cdot cd} + f^{ab\cdot}f^{\cdot c\cdot}f^{\cdot ed} + f^{ae\cdot}f^{\cdot d\cdot}f^{\cdot cb} + f^{ab\cdot}f^{\cdot e\cdot}f^{\cdot cd}] \\
&\quad + \mathrm{i}\frac{C_2}{60}[f^{bd\cdot}f^{\cdot a\cdot}f^{\cdot ce} + f^{ad\cdot}f^{\cdot b\cdot}f^{\cdot ce} + f^{ad\cdot}f^{\cdot c\cdot}f^{\cdot be\cdot} + f^{ac\cdot}f^{\cdot d\cdot}f^{\cdot be} + f^{ac\cdot}f^{\cdot e\cdot}f^{\cdot bd}],
\end{aligned}$$

$$\begin{aligned}
\mathrm{Sp}\, & X^k\underbrace{X^aX^b}_{}X^k\underbrace{X^cX^d}_{} \\
&= \mathrm{Sp}\, X^aX^bX^cX^d \cdot \frac{C_2}{3} + \mathrm{Sp}\, X^bX^aX^cX^d \cdot \frac{C_2}{3}\left(-\frac{1}{2}\right) \\
&\quad + \mathrm{Sp}[X^a, X^c]X^dX^b \cdot \frac{C_2}{3}\left(\frac{5}{N+2}\right) + \mathrm{Sp}[X^b, X^c]X^dX^a \cdot \frac{C_2}{3}\left(\frac{5}{N+2}\right) \\
&\quad + \mathrm{Sp}\, X^aX^b\, \mathrm{Sp}\, X^cX^d \cdot \frac{C_2}{6}\left(\frac{5}{N+2}\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Sp}\, & X^k\underbrace{X^aX^bX^c}_{}X^k\underbrace{X^dX^e}_{} \\
&= \mathrm{Sp}\, X^aX^bX^cX^dX^e \cdot \frac{C_2}{3} + \mathrm{Sp}\, X^aX^bX^cX^eX^d \cdot \frac{C_2}{3}\left(-\frac{1}{2}\right) \\
&\quad + \mathrm{Sp}[X^d, X^a]X^b[X^c, X^e] \cdot \frac{C_2}{3}\left(\frac{5}{N+2}\right) \\
&\quad + \mathrm{Sp}[X^e, X^a]X^b[X^c, X^d] \cdot \frac{C_2}{3}\left(\frac{5}{N+2}\right) \\
&\quad + \mathrm{Sp} X^aX^bX^c\, \mathrm{Sp}\, X^dX^e \cdot \frac{C_2}{6}\left(\frac{5}{N+2}\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Sp}\, & X^7\underbrace{X^aX^bX^cX^d}_{}X^k\underbrace{X^eX^f}_{} \\
&= \mathrm{Sp}\, X^aX^bX^cX^dX^eX^f \cdot \frac{C_2}{3} + \mathrm{Sp}\, X^aX^bX^cX^dX^fX^e \cdot \frac{C_2}{3}\left(-\frac{1}{2}\right)
\end{aligned}$$

$$+\mathrm{Sp}\,[X^e, X^a]X^b[X^c, [X^d, X^f]]\cdot\frac{C_2}{3}\left(\frac{5}{N+2}\right)$$

$$+\mathrm{Sp}[X^f, X^a]X^b[X^c, [X^d, X^e]]\cdot\frac{C_2}{3}\left(\frac{5}{N+2}\right)$$

$$+\mathrm{Sp}\,X^aX^bX^cX^d\,\mathrm{Sp}\,X^eX^f\cdot\frac{C_2}{6}\left(\frac{5}{N+2}\right).$$

$$\mathrm{Sp}\,X^k\underbracket{X^aX^bX^c}X^k\underbracket{X^dX^eX^f}$$

$$=\mathrm{Sp}\,X^aX^bX^cX^dX^eX^f\cdot\frac{C_2}{3}$$

$$+\mathrm{Sp}\,X^cX^bX^aX^dX^eX^f\cdot\frac{C_2}{3}\left(\frac{1}{2}\right)$$

$$\left.\begin{aligned}&+\mathrm{Sp}[X^a, X^c]X^d[X^e, [X^f, X^b]]\\&+\mathrm{Sp}[X^b, X^c]X^d[X^e, [X^f, X^a]]\\&+\mathrm{Sp}[X^a, X^b]X^d[X^e, [X^f, X^c]]\\&+\mathrm{Sp}[X^b, X^d]X^e(X^f, [X^a, X^c]]\\&+\mathrm{Sp}[X^a, X^d]X^e[X^f, [X^b, X^c]]\\&+\mathrm{Sp}[X^c, X^d]X^e[X^f, [X^a, X^b]]\end{aligned}\right\}\frac{C_2}{6}\left(\frac{5}{N+2}\right)$$

$$+\mathrm{Sp}\,X^aX^bX^c\,\mathrm{Sp}\,X^dX^eX^f\cdot\frac{C_2}{6}\left(\frac{5}{N+2}\right).$$

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